

Sign changing solutions for quasilinear superlinear elliptic problems

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Abstract Results on existence and multiplicity of solutions for a nonlinear elliptic problem driven by the Φ -Laplace operator are established. We employ minimization arguments on suitable Nehari manifolds to build a negative and a positive ground state solutions. In order to find a nodal solution we employ additionally the well known Deformation Lemma and Topological Degree Theory.

Keywords Variational methods · Quasilinear Elliptic Problems · Nehari manifold method

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1 Introduction

In this work we consider the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function which is C^1 in the second variable. For the function ϕ we assume that $\phi : (0, \infty) \rightarrow (0, \infty)$ is of class C^2 and satisfies the following conditions:

$$(\phi_1) \quad \lim_{t \rightarrow 0} t\phi(t) = 0, \quad \lim_{t \rightarrow \infty} t\phi(t) = \infty;$$

$$(\phi_2) \quad t \mapsto t\phi(t) \text{ is strictly increasing.}$$

We point out that the function $\phi(t) = t^{p-2}$ for $t > 0$, with $1 < p < \infty$, satisfies $(\phi_1) - (\phi_2)$ and in this case the operator in problem (1.1) is named p -Laplacian and (1.1) reads as

$$-\Delta_p u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

In a similar way, the function $\phi(t) = t^{p-2} + t^{q-2}$ with $1 < q < p < \infty$ satisfies the conditions $(\phi_1) - (\phi_2)$. In this case the operator in problem (1.1) is named (p, q) -Laplacian and problem (1.1) becomes

$$-\Delta_p u - \Delta_q u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

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In the model case $f(s) = |s|^{p-2}s$ it is well-known that the Ambrosetti-Rabinowitz condition (see [2]), (AR) for short, namely

$$\begin{aligned} & \text{there exist } \theta > 2, R > 0 \text{ such that} \\ & 0 < \theta F(x, t) \leq t f(x, t), x \in \Omega, |t| \geq R, \end{aligned} \quad (AR)$$

with $F(x, t) = \int_0^t f(x, s)ds$, plays a crucial role while addressing compactness requirement in variational methods. However, there exist lots of functions for which (AR) is not satisfied. For instance, $f(t) = t \log(1 + |t|)$ for $t \in \mathbb{R}$ does not satisfy (AR). It is important to emphasize that the main role of (AR) is to ensure the well known (PS)-condition required by minimax arguments. We refer the reader to the reasearch papers [9, 19, 21, 20, 22, 23, 30] and references therein, where problems involving the p -Laplacian, sometimes with $p = 2$, and the (p, q) -Laplacian operator have been addressed.

There is a rich literature on problems of the form (1.1) with functions ϕ even more general than the ones mentioned above. In such more general settings the operator in (1.1) is named Φ -Laplacian and is written as

$$\operatorname{div}(\phi(|\nabla u|)\nabla u) := \Delta_\Phi u,$$

with

$$\Phi(t) = \int_0^t s\phi(s)ds, t \in \mathbb{R},$$

where $0 < s \rightarrow s\phi(s)$ has been extended to the whole \mathbb{R} as an odd function and so Φ is an even function. In [10], Clément, García-Huidobro, Manásevich & Schmitt showed results for problems of the form

$$-\Delta_\Phi u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where the nonlinear term f satisfies (AR). In the recent paper [7], results on existence and multiplicity of solutions were proven.

We refer the reader to Radulescu [26] and its references where operators even more general than the Φ -Laplacian are treated, and motivation from the physical sciences are discussed.

Due to the nature of the operator Δ_Φ we shall work in the framework of Orlicz-Sobolev spaces $W_0^{1,\Phi}(\Omega)$. Some basic facts and references on these spaces are given in section 2.

We shall assume the following condition on ϕ :

$$(\phi_3) \quad -1 < \ell - 2 := \inf_{t>0} \frac{(t\phi(t))''t}{(t\phi(t))'} \leq \sup_{t>0} \frac{(t\phi(t))''t}{(t\phi(t))'} := m - 2 < N - 2.$$

Remark 1.1. *It can be shown that (ϕ_3) implies the (less restrictive) condition:*

$$(\phi_3)' \quad 1 < \ell := \inf_{t>0} \frac{t^2\phi(t)}{\Phi(t)} \leq \sup_{t>0} \frac{t^2\phi(t)}{\Phi(t)} =: m < N,$$

Moreover under conditions $(\phi_1), (\phi_2), (\phi_3)'$ the space $W_0^{1,\Phi}(\Omega)$ is a reflexive Banach space, (see Remark 2.1 at Section 2 ahead).

We denote by $\lambda_1 > 0$ the first eigenvalue for the operator $-\Delta_\Phi$. Recall that it satisfies the Poincaré inequality, (see e.g. [10], [17]),

$$\lambda_1 \int_\Omega \Phi(u)dx \leq \int_\Omega \Phi(|\nabla u|)dx, \quad u \in W_0^{1,\Phi}(\Omega). \quad (1.2)$$

The following conditions will be imposed on the nonlinear term f :

(f_0) there are a function $\psi : [0, \infty) \rightarrow [0, \infty)$ and a constant $C > 0$ such that

$$|f(x, t)| \leq C [1 + \psi(t)], \quad t \in \mathbb{R}, \quad x \in \Omega,$$

where $\Psi(t) = \int_0^t \psi(s) ds$ is an N -function satisfying $\Psi \ll \Phi_*$, (definitions and properties in Section 2), and

$$(\psi_1) \quad 1 < \ell \leq m < \ell_\Psi := \inf_{t>0} \frac{t\psi(t)}{\Psi(t)} \leq \sup_{t>0} \frac{t\psi(t)}{\Psi(t)} =: m_\Psi < \ell^* := \frac{\ell N}{N - \ell};$$

(f_1) the function

$$t \mapsto \frac{f(x, t)}{|t|^{m-2}t}$$

is increasing on $\mathbb{R} \setminus \{0\}$;

(f_2) the limit

$$\lim_{t \rightarrow 0} \frac{f(x, t)}{t\phi(t)} < \lambda_1$$

holds uniformly in $x \in \Omega$;

(f_3) the limit

$$\lim_{|t| \rightarrow \infty} \frac{f(x, t)}{|t|^{m-2}t} = +\infty$$

holds uniformly for $x \in \Omega$.

The key technique in the proof of the main result of the present paper will be the use of the Nehari manifold method, (see the pioneering work [24]), this time for the energy functional J associated with problem (1.1) which due to the nature of the operator Δ_Φ is defined in an Orlicz-Sobolev space. Here one of the main difficulties is to ensure that a minimizing sequence for J over the Nehari manifold converges to a critical point. Another difficulty is to show that the Nehari manifold is C^1 . To overcome these difficulties we prove that the map $u \mapsto \langle J'(u), u \rangle$ is of class C^1 . Condition (ϕ_3) is crucial to achieve these steps. We will also make use of some Topological Degree arguments.

The main result in this work is:

Theorem 1.1. *Assume $(\phi_1), (\phi_2), (\phi_3), (f_0) - (f_3)$. Then problem (1.1) admits at least two ground state solutions $u_1, u_2 \in W_0^{1, \Phi}(\Omega)$ satisfying $u_1 < 0$ and $u_2 > 0$ in Ω . In addition, problem (1.1) admits a further solution say u_3 which changes sign in Ω .*

To our best knowledge, there is no result on existence of sign changing solutions for problems involving the Φ -Laplacian operator.

Existence of positive and negative solutions have long been searched for problems involving both Laplacian and p -Laplacian equations. We would like to mention the works [7], [10], [9], [14], [15],

[18], [30], [32] and references therein which are more related to our present interest in this paper. In those works the authors have used truncation techniques and maximum principles.

More recently, sign changing solutions have been considered. We refer the reader to Szulkin and Weth [29, 28], where the authors also addressed existence of positive solutions for the Dirichlet problem for Laplace and p-Laplace equations using the Nehari manifold method.

We also refer the reader to [4], [3], [13] and [31]. In these works the authors considered semilinear/superlinear problems driven by the Laplace and the p-Laplace operators. In [3] the authors considered quasilinear problems for the p-Laplacian operator obtaining existence of positive solutions. The results in the present paper for the case of the Φ -Laplace operator complement/extend ones in the above-mentioned papers.

We also point out that quasilinear elliptic problems have been considered under several assumptions on the nonlinear term f . In this regard we refer the reader to [7, 9, 17, 19, 30, 8]. In [9], [19] the authors considered monotonicity conditions on the nonlinear term f , proving existence of positive solutions. In [7], [19] the authors studied quasilinear elliptic problems without the Ambrosetti-Rabinowitz condition at infinity obtaining existence of multiple solutions.

Remark 1.2. *An example of a problem to which our theorem 1.1 applies is*

$$-\Delta_{\Phi} u = pu^{p-1} \log(1+u) + \frac{u^p}{u+1} \text{ in } \Omega, u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where

$$\Phi(t) = |t|^{\gamma} \log(1+|t|) \text{ with } 1 < \frac{-1 + \sqrt{1+4N}}{2} < \gamma < N-1,$$

$$F(t) = t^p \log(1+|t|) \text{ and } F'(t) = pt^{p-1} \log(1+t) + \frac{t^p}{t+1} := f(t), \quad t > 0,$$

with $\gamma = \ell$, $\gamma + 1 = m$, $\sigma > \frac{N}{\ell}$ and $m < p < \frac{\ell\sigma}{\sigma-1}$. We notice that by an easy computation,

$$t\phi(t) = \Phi'(t) = \gamma t^{\gamma-1} \log(1+t) + \frac{t^{\gamma}}{1+t}, \quad t > 0$$

and the conditions $(\phi_1), (\phi_2), (\phi_3)$ are satisfied. In addition, the function $f(t)$ satisfies $(f_0) - (f_3)$.

The operator Δ_{Φ} in the present example appears in Plasticity, see e.g. Fukagai and Narukawa [15].

This paper is organized as follows: in section 2 we recall some basic properties of Orlicz-Sobolev spaces. Section 3 is devoted to auxiliary results on functionals defined on Orlicz-Sobolev spaces and related Nehari manifolds. In Section 4 we give the proof of the main result of Section 3, namely Theorem 3.1 which ensures existence of a ground state solution of problem (1.1). Finally Section 5 is devoted to the proof of Theorem 1.1.

2 Basics on Orlicz-Sobolev spaces

The reader is referred to [1, 27] regarding Orlicz-Sobolev spaces. The usual norm on $L_{\Phi}(\Omega)$ (Luxemburg norm) is ,

$$\|u\|_{\Phi} = \inf \left\{ \lambda > 0 \mid \int_{\Omega} \Phi \left(\frac{u(x)}{\lambda} \right) dx \leq 1 \right\}$$

and the Orlicz-Sobolev norm of $W^{1,\Phi}(\Omega)$ is

$$\|u\|_{1,\Phi} = \|u\|_{\Phi} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{\Phi}.$$

Recall that

$$\tilde{\Phi}(t) = \max_{s \geq 0} \{ts - \Phi(s)\}, \quad t \geq 0.$$

Remark 2.1. We notice that, $(\phi_1), (\phi_2), (\phi_3)'$ imply that Φ and $\tilde{\Phi}$ are N -functions satisfying the Δ_2 -condition. In addition, $L_\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable, reflexive, Banach spaces, (cf. [27]). We recall that (ϕ_3) implies $(\phi_3)'$.

Using the Poincaré inequality (1.2) it follows that

$$\|u\|_\Phi \leq C \|\nabla u\|_\Phi \text{ for each } u \in W_0^{1,\Phi}(\Omega).$$

holds true for some $C > 0$. As a consequence, $\|u\| := \|\nabla u\|_\Phi$ defines a norm in $W_0^{1,\Phi}(\Omega)$, equivalent to $\|\cdot\|_{1,\Phi}$. Let Φ_* be the inverse of the function

$$t \in (0, \infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds$$

which extends to \mathbb{R} by $\Phi_*(t) = \Phi_*(-t)$ for $t \leq 0$. We say that an N -function Ψ grows essentially more slowly than Φ_* , we write $\Psi << \Phi_*$, if

$$\lim_{t \rightarrow \infty} \frac{\Psi(\lambda t)}{\Phi_*(t)} = 0, \quad \text{for all } \lambda > 0.$$

The imbedding below (cf. [1, 12]) will be used in this paper:

$$W_0^{1,\Phi}(\Omega) \xrightarrow{cpt} L_\Psi(\Omega) \quad \text{if } \Psi << \Phi_*,$$

in particular, as $\Phi << \Phi_*$ (cf. [16, Lemma 4.14]),

$$W_0^{1,\Phi}(\Omega) \xrightarrow{cpt} L_\Phi(\Omega).$$

Furthermore,

$$W_0^{1,\Phi}(\Omega) \xrightarrow{\text{cont}} L_{\Phi_*}(\Omega).$$

Remark 2.2. The condition (ψ_1) shows that $\Psi << \Phi_*$, i.e., the function Ψ grows essentially more slowly than Φ_* . In fact, by Proposition 2.1, stated below,

$$\lim_{t \rightarrow \infty} \frac{\Psi(\lambda t)}{\Phi_*(t)} \leq \frac{\lambda^{m_\Psi}}{\Phi_*(1)} \lim_{t \rightarrow \infty} \frac{1}{t^{\ell^* - m_\Psi}} = 0, \quad \text{for all } \lambda > 0.$$

In this case $W_0^{1,\Phi}(\Omega) \xrightarrow{cpt} L_\Psi(\Omega)$.

We refer the reader to [14] for the two results below.

Proposition 2.1. Assume that ϕ satisfies $(\phi_1), (\phi_2), (\phi_3)'$. Set

$$\zeta_0(t) = \min\{t^\ell, t^m\}, \quad \zeta_1(t) = \max\{t^\ell, t^m\}, \quad t \geq 0.$$

Then Φ satisfies

$$\begin{aligned} \zeta_0(t)\Phi(\rho) &\leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho), \quad \rho, t > 0, \\ \zeta_0(\|u\|_\Phi) &\leq \int_\Omega \Phi(u) dx \leq \zeta_1(\|u\|_\Phi), \quad u \in L_\Phi(\Omega). \end{aligned}$$

Proposition 2.2. Assume that ϕ satisfies $(\phi_1), (\phi_2), (\phi_3)'$. Set

$$\zeta_2(t) = \min\{t^{\ell^*}, t^{m^*}\}, \quad \zeta_3(t) = \max\{t^{\ell^*}, t^{m^*}\}, \quad t \geq 0$$

where $1 < \ell, m < N$ and $m^* = \frac{mN}{N-m}$, $\ell^* = \frac{\ell N}{N-\ell}$. Then

$$\ell^* \leq \frac{t^2 \Phi'_*(t)}{\Phi_*(t)} \leq m^*, \quad t > 0,$$

$$\zeta_2(t) \Phi_*(\rho) \leq \Phi_*(\rho t) \leq \zeta_3(t) \Phi_*(\rho), \quad \rho, t > 0,$$

$$\zeta_2(\|u\|_{\Phi_*}) \leq \int_{\Omega} \Phi_*(u) dx \leq \zeta_3(\|u\|_{\Phi_*}), \quad u \in L_{\Phi_*}(\Omega).$$

3 Nehari manifolds in Orlicz-Sobolev spaces and Ground State Solutions of problem (1.1)

Under the conditions of the present paper the energy functional J of (1.1) given by

$$J(u) = \int_{\Omega} \Phi(|\nabla u|) dx - \int_{\Omega} F(x, u) dx, \quad u \in W_0^{1,\Phi}(\Omega),$$

where

$$F(x, t) = \int_0^t f(x, s) ds \text{ for } s \in \mathbb{R},$$

is of class C^1 and actually

$$\langle J'(u), v \rangle = \int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v \, dx - \int_{\Omega} f(x, u) v \, dx, \quad u, v \in W_0^{1,\Phi}(\Omega).$$

Finding weak solutions of problem (1.1) is equivalent to find critical points of J .

The Nehari manifold associated to J is given by

$$\mathcal{N} = \{u \in W_0^{1,\Phi}(\Omega) \setminus \{0\} \mid \langle J'(u), u \rangle = 0\}.$$

The main result of this section is:

Theorem 3.1. Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Then problem (1.1) admits a nonzero ground state solution u in the sense that $u \in \mathcal{N}$,

$$\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v \, dx = \int_{\Omega} f(x, u) v \, dx, \quad v \in W_0^{1,\Phi}(\Omega), \text{ and}$$

$$J(u) = \inf_{w \in \mathcal{N}} J(w).$$

The proof of Theorem 3.1 will be given in Section 4.

Initially we will establish and prove a few technical results.

Proposition 3.1. Assume $(\phi_1) - (\phi_3)$ and (f_0) . Then the functionals

$$(i) \quad u \in W_0^{1,\Phi}(\Omega) \mapsto \int_{\Omega} F(x, u) dx, \quad (ii) \quad u \in W_0^{1,\Phi}(\Omega) \mapsto \int_{\Omega} f(x, u) u dx$$

are weakly sequentially continuous, w.s.c. for short.

Proof of Proposition 3.1. The compact embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Phi(\Omega)$ and $W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Psi(\Omega)$ will play a crucial role. Remind that $\Phi \ll \Phi_*$ and $\Psi \ll \Phi_*$.

Let (u_n) be a sequence in $W_0^{1,\Phi}(\Omega)$ such that $u_n \rightharpoonup u$ for some $u \in W_0^{1,\Phi}(\Omega)$. Hence, up to a subsequence, $u_n \rightarrow u$ in $L_\Psi(\Omega)$, $u_n \rightarrow u$ a.e. in Ω and $|u_n| \leq h$ for some $h \in L_\Psi(\Omega)$. Consequently, using (f_0) , we obtain

$$|f(x, u_n)u_n| \leq C|u_n| + C\Psi(u_n) \leq Ch + C\Psi(h) \in L^1(\Omega).$$

By the Dominated Convergence Theorem we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x, u_n)u_n dx = \int_{\Omega} f(x, u)u dx.$$

Similarly, one shows that

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(x, u_n) dx = \int_{\Omega} F(x, u) dx.$$

This finishes the proof. ■

Remark 3.1. We gather some facts about the behavior of F both near the origin and at infinity. Let $\varepsilon > 0$ be a small number. By (f_0) and (f_2) there is a positive constant C_ε such that

$$|f(x, t)| \leq (\lambda_1 - \varepsilon)|t\phi(t)| + C_\varepsilon\psi(t), \quad t \in \mathbb{R}$$

and

$$|F(x, t)| \leq (\lambda_1 - \varepsilon)\Phi(t) + C_\varepsilon\Psi(t), \quad t \in \mathbb{R}. \quad (3.1)$$

In addition, using (f_2) and $(\phi_3)'$ one finds that

$$\limsup_{t \rightarrow 0} \frac{tf(x, t)}{\Phi(t)} < \frac{\lambda_1}{m}. \quad (3.2)$$

By (f_0) and (3.2) it follows that

$$|tf(x, t)| \leq \left(\frac{\lambda_1 - \varepsilon}{m} \right) \Phi(t) + C_\varepsilon\Psi(t), \quad t \in \mathbb{R}. \quad (3.3)$$

As a consequence of (3.1) and (3.3), for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that for each $u \in W_0^{1,\Phi}(\Omega)$,

$$\begin{aligned} \int_{\Omega} f(x, u)u dx &\leq \left(\frac{\lambda_1 - \varepsilon}{m} \right) \int_{\Omega} \Phi(u) dx + C_\varepsilon \int_{\Omega} \Psi(u) dx, \\ \int_{\Omega} F(x, u) dx &\leq (\lambda_1 - \varepsilon) \int_{\Omega} \Phi(u) dx + C_\varepsilon \int_{\Omega} \Psi(u) dx. \end{aligned} \quad (3.4)$$

Using the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Psi(\Omega)$ and Proposition 2.1 it follows that

$$\int_{\Omega} f(x, u)u dx \leq \left(\frac{\lambda_1 - \varepsilon}{m} \right) \int_{\Omega} \Phi(u) dx + C_\varepsilon \max(\|u\|^{\ell_\Psi}, \|u\|^{m_\Psi})$$

and

$$\int_{\Omega} F(x, u) dx \leq (\lambda_1 - \varepsilon) \int_{\Omega} \Phi(u) dx + C_\varepsilon \max(\|u\|^{\ell_\Psi}, \|u\|^{m_\Psi}) \quad (3.5)$$

for $u \in W_0^{1,\Phi}(\Omega)$.

At this point, aiming to determine the behavior of J on \mathcal{N} we introduce the fibering maps $\gamma_u : (0, \infty) \rightarrow \mathbb{R}$ for $u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$, defined by

$$\gamma_u(t) = J(tu), \quad t \in (0, \infty),$$

(see [5, 6]). In this regard we shall study the behavior of $\gamma_u(t)$ for both t near infinity and t near the origin.

Our next result is:

Proposition 3.2. *Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$ and let $u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$. Then*

$$(i) \quad \lim_{t \rightarrow 0} \frac{\gamma_u(t)}{t^m} > 0, \quad \lim_{t \rightarrow \infty} \frac{\gamma_u(t)}{t^m} = -\infty,$$

and in addition,

$$(ii) \quad \lim_{t \rightarrow 0} \frac{\gamma'_u(t)}{t^{m-1}} > 0, \quad \lim_{t \rightarrow \infty} \frac{\gamma'_u(t)}{t^{m-1}} = -\infty.$$

Proof of Proposition 3.2. By (3.5) one infers that

$$\gamma_u(t) \geq \int_{\Omega} \Phi(|\nabla(tu)|) dx - (\lambda_1 - \varepsilon) \int_{\Omega} \Phi(tu) dx - C_{\varepsilon} \max \left(\|tu\|^{\ell_{\Psi}}, \|(tu)\|^{m_{\Psi}} \right).$$

Applying the Poincaré inequality we have

$$\gamma_u(t) \geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \int_{\Omega} \Phi(|\nabla tu|) dx - C_{\varepsilon} \max \left(\|tu\|^{\ell_{\Psi}}, \|tu\|^{m_{\Psi}} \right).$$

Applying Proposition 2.1 it follows that

$$\gamma_u(t) \geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) t^m \int_{\Omega} \Phi(|\nabla u|) dx - C_{\varepsilon} \max \left(\|tu\|^{\ell_{\Psi}}, \|tu\|^{m_{\Psi}} \right).$$

By the arguments above we have: for each $u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$ and $0 < t < 1$,

$$\frac{\gamma_u(t)}{t^m} \geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \int_{\Omega} \Phi(|\nabla u|) dx - C_{\varepsilon} \frac{\max \left(\|tu\|^{\ell_{\Psi}}, \|tu\|^{m_{\Psi}} \right)}{t^m}. \quad (3.6)$$

Using the fact that $m < \ell_{\Psi}$, (3.6) rewrites as

$$\frac{\gamma_u(t)}{t^m} \geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \int_{\Omega} \Phi(|\nabla u|) dx + o(1)$$

where $o(1)$ denotes a quantity that goes to zero as $t \rightarrow 0$.

Hence

$$\lim_{t \rightarrow 0} \frac{\gamma_u(t)}{t^m} \geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \int_{\Omega} \Phi(|\nabla u|) dx > 0.$$

Next we shall compute the limit of $\gamma_u(t)/t^m$ at infinity. Using Proposition 2.1 we get

$$\frac{\gamma_u(t)}{t^m} \leq \int_{\Omega} \Phi(|\nabla u|) dx - \frac{1}{t^m} \int_{\Omega} F(x, tu) dx.$$

Applying Fatou's Lemma and (f_3) we get

$$\lim_{t \rightarrow \infty} \frac{\gamma_u(t)}{t^m} \leq \int_{\Omega} \Phi(|\nabla u|) dx - \liminf_{t \rightarrow \infty} \int_{\Omega} \frac{F(x, tu)}{t^m} dx = -\infty.$$

We emphasize that (f_0) and (f_3) ensure that

$$\liminf_{t \rightarrow \infty} \int_{\Omega} \frac{F(x, tu)}{t^m} dx = \liminf_{t \rightarrow \infty} \int_{\Omega} \frac{F(x, tu)}{|tu|^m} |u|^m dx = +\infty.$$

The limits involving $\gamma'_u(t)/t^{m-1}$ are computed by arguments similar to the ones above. This ends the proof. \blacksquare

Proposition 3.3. *Suppose $(\phi_1), (\phi_2), (\phi_3)$. Then the functionals*

$$(i) \quad u \in W_0^{1,\Phi}(\Omega) \mapsto \int_{\Omega} \phi(|\nabla u|) |\nabla u|^2 dx$$

and

$$(ii) \quad u \in W_0^{1,\Phi}(\Omega) \mapsto \int_{\Omega} (m\Phi(|\nabla u|) - \phi(|\nabla u|) |\nabla u|^2) dx$$

are weakly sequentially lower semicontinuous, w.s.l.s.c. for short.

Proof of Proposition 3.3. It is an easy matter to show that both functionals are lower semicontinuous. So it is enough to show that they are also convex. To this end consider the functions

$$L_1(t) = \phi(t)t^2, \quad L_2(t) = m\Phi(t) - \phi(t)t^2, \quad t \geq 0.$$

It follows by an easy computation, using condition (ϕ_3) that

$$L_1''(t) \geq 2(t\phi(t))' + (\ell - 2)(t\phi(t))' = l(t\phi(t))' \geq 0.$$

On the other hand, this time using condition $(\phi_3)'$ we infer that

$$L_2''(t) \geq (m - \ell)(t\phi(t))' \geq 0.$$

Thus L_1, L_2 are convex functions. As a consequence the functionals in (i)-(ii) above are convex. This finishes the proof. \blacksquare

Next we shall discuss the relation between the fibering map γ_u and \mathcal{N} . Roughly speaking it will be shown that the map γ_u crosses \mathcal{N} once in a suitable sense.

Proposition 3.4. *Assume $(\phi_1) - (\phi_3), (f_0) - (f_3)$. Then for each $u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$ there is only a $t = t(u) > 0$ such that $tu \in \mathcal{N}$. Moreover, $J(u) > 0$ for each $u \in \mathcal{N}$.*

Proof of Proposition 3.4. Let $u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$. At first, note that by the very definition of $\gamma_u(t)$, $tu \in \mathcal{N}$ if and only if $\gamma'_u(t) = 0$.

On the other hand, by Proposition 3.2 we have

$$\gamma'_u(t) > 0 \quad \text{for } t \text{ small enough}$$

and

$$\gamma'_u(t) < 0 \quad \text{for } t \text{ big enough.}$$

Since the map $t \mapsto \gamma'_u(t)$ is continuous there is at least one number $t \in (0, \infty)$ such that $\gamma'_u(t) = 0$. This means that $tu \in \mathcal{N}$.

We claim that there is only one $t = t(u)$ such that $\gamma'_u(t) = 0$. Indeed, recall that $\gamma'_u(t) = \langle J'(tu), u \rangle$. So

$$\frac{d}{dt} \left[\frac{\gamma'_u(t)}{t^{m-1}} \right] = \int_{\Omega} \frac{d}{dt} \left[\frac{\phi(|\nabla tu|) \nabla tu \nabla u}{t^{m-1}} \right] dx - \int_{\Omega} \frac{d}{dt} \left[\frac{f(x, tu)u}{t^{m-1}} \right] dx. \quad (3.7)$$

At this point we remark that (ϕ_3) implies

$$\ell - 2 \leq \inf_{t>0} \frac{t\phi'(t)}{\phi(t)} \leq \sup_{t>0} \frac{t\phi'(t)}{\phi(t)} \leq m - 2.$$

Using the inequalities just above we infer that for each $x \in \Omega$ and $t > 0$,

$$\frac{d}{dt} \left[\frac{\phi(|\nabla tu|) \nabla tu \nabla u}{t^{m-1}} \right] = \frac{|\nabla u|^2 [\phi'(|\nabla tu|) |\nabla tu| - (m-2)\phi(|\nabla tu|)]}{t^{m-1}} \leq 0. \quad (3.8)$$

It follows by using (3.7) and (3.8) that

$$\frac{d}{dt} \left[\frac{\gamma'_u(t)}{t^{m-1}} \right] \leq - \int_{\Omega} \frac{d}{dt} \left[\frac{f(x, tu)u}{t^{m-1}} \right] dx = - \int_{\Omega} \frac{d}{dt} \left[\frac{f(x, tu)}{|tu|^{m-2} tu} \right] |u|^m dx. \quad (3.9)$$

Now, using (f_1) and (3.9) we get

$$\frac{d}{dt} \left[\frac{\gamma'_u(t)}{t^{m-1}} \right] < 0$$

for each $t > 0$ and $u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$.

Therefore $t \mapsto \frac{\gamma'_u(t)}{t^{m-1}}$ is a decreasing function that vanishes once in $(0, \infty)$ so that there is an only $t = t(u) > 0$ such that $\gamma'_u(t)/t^{m-1} = 0$.

Thus the function γ_u admits a unique critical point namely $t = t(u) > 0$ and actually, $tu \in \mathcal{N}$. Moreover it follows by Proposition 3.2 that $t(u)$ is a maximum point of γ_u on $(0, \infty)$ and, in fact $\gamma_u(t(u)) > 0$, which implies that $J(t(u)u) > 0$. The arguments above also show that $\gamma''_u(t) < 0$ for each $u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$.

Finally, since $u \in \mathcal{N}$ if and only if $t(u) = 1$, we deduce that $J(u) > 0$ for each $u \in \mathcal{N}$. This completes the proof. \blacksquare

Proposition 3.5. *Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Then $I : W_0^{1,\Phi}(\Omega) \rightarrow \mathbf{R}$,*

$$I(u) = \int_{\Omega} \phi(|\nabla u|) |\nabla u|^2 dx, \quad u \in W_0^{1,\Phi}(\Omega),$$

is C^1 and

$$\langle I'(u), v \rangle = \int_{\Omega} [2\phi(|\nabla u|) + \phi'(|\nabla u|) |\nabla u|] \nabla u \cdot \nabla v dx, \quad u, v \in W_0^{1,\Phi}(\Omega).$$

Proof of Proposition 3.5. Set

$$g(t) = \phi(|\nabla u + t \nabla v|) |\nabla u + t \nabla v|^2, \quad 0 \leq t \leq 1.$$

It follows that $g \in C^1$ and, actually

$$g'(t) = [2\phi(|\nabla u + t \nabla v|) + \phi'(|\nabla u + t \nabla v|) |\nabla u + t \nabla v|] (\nabla u + t \nabla v) \cdot \nabla v.$$

In addition, there is $\theta \in \mathbf{R}$ with $0 < \theta < t \leq 1$ such that

$$\frac{g(t) - g(0)}{t} = g'(\theta).$$

Thus

$$\begin{aligned} \langle I'(u), v \rangle &= \lim_{t \rightarrow 0} \int_{\Omega} \frac{\phi(|\nabla u + t\nabla v|)|\nabla u + t\nabla v|^2 - \phi(|\nabla u|)|\nabla u|^2}{t} dx \\ &= \lim_{\theta \rightarrow 0} \int_{\Omega} g'(\theta) dx. \end{aligned}$$

Claim. There is $h \in L^1(\Omega)$ such that $|g'(\theta)| \leq h$.

At first we recall that by $(\phi_3)'$,

$$|\phi'(t)t| \leq \max\{|\ell - 2|, |m - 2|\}\phi(t), \quad 0 \leq t < \infty \quad (3.10)$$

and as a consequence,

$$\lim_{t \rightarrow 0} |\phi'(t)t^2| = \lim_{t \rightarrow 0} [\max\{|\ell - 2|, |m - 2|\}\phi(t)t] = 0. \quad (3.11)$$

Using $(\phi_1), (\phi_2), (\phi_3)$ and $0 \leq \theta \leq 1$ we have,

$$\begin{aligned} |g'(\theta)| &\leq [2\phi(|\nabla u + \theta\nabla v|) + |\phi'(|\nabla u + \theta\nabla v|)|\nabla u + \theta\nabla v|]|\nabla u + \theta\nabla v||\nabla v| \\ &\leq [2 + \max\{|\ell - 2|, |m - 2|\}]\phi(|\nabla u| + |\nabla v|)(|\nabla u| + |\nabla v|)|\nabla v|. \end{aligned}$$

Next we show that $[2 + \max\{|\ell - 2|, |m - 2|\}]\phi(|\nabla u| + |\nabla v|)(|\nabla u| + |\nabla v|)|\nabla v| \in L^1(\Omega)$.

Indeed, using Young's inequality, the inequality $\tilde{\Phi}(t\phi(t)) \leq \Phi(2t)$ and the fact that $\Phi \in \Delta_2$ we have

$$\begin{aligned} \phi(|\nabla u| + |\nabla v|)(|\nabla u| + |\nabla v|)|\nabla v| &\leq \Phi(|\nabla v|) + \tilde{\Phi}(\phi(|\nabla u| + |\nabla v|)(|\nabla u| + |\nabla v|)) \\ &\leq \Phi(|\nabla v|) + \Phi(2(|\nabla u| + |\nabla v|)) \\ &\leq \Phi(|\nabla v|) + 2^m \Phi(|\nabla u| + |\nabla v|) \\ &\leq (1 + 2^m)\Phi(|\nabla u| + |\nabla v|) \in L^1(\Omega). \end{aligned}$$

So,

$$|g'(\theta)| \leq [2 + \max\{|\ell - 2|, |m - 2|\}](1 + 2^m)\Phi(|\nabla u| + |\nabla v|) := h \in L^1(\Omega).$$

This ends the proof of the claim.

It remains to show that I' is continuous. Indeed, let $(u_n) \subseteq W_0^{1,\Phi}(\Omega)$ such that $u_n \rightarrow u$ in $W_0^{1,\Phi}(\Omega)$.

Then

$$\int_{\Omega} \Phi(|\nabla u_n - \nabla u|) dx \xrightarrow{n \rightarrow \infty} 0,$$

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e. in } \Omega,$$

$$|\nabla u_n| \leq h_1 \quad \text{a.e. in } \Omega \text{ for some } h_1 \in L^1(\Omega).$$

By arguments as above,

$$\begin{aligned}
|[2\phi(|\nabla u_n|) + \phi'(|\nabla u_n|)|\nabla u_n|]\nabla u_n \cdot \nabla v| &\leq [2\phi(|\nabla u_n|) + |\phi'(|\nabla u_n|)||\nabla u_n|]\nabla u_n \cdot \nabla v \\
&\leq [2\phi(|\nabla u_n|)|\nabla u_n| + |\phi'(|\nabla u_n|)||\nabla u_n|^2]|\nabla v| \\
&\leq C\phi(|\nabla u_n|)|\nabla u_n||\nabla v| \\
&\leq C]\phi(h_1)h_1|\nabla v| \in L^1(\Omega),
\end{aligned}$$

where $C := 2 + \max(|\ell - 2|, |m - 2|)$.

Since $\nabla u_n \rightarrow \nabla u$ a.e. in Ω we get by (3.10)-(3.11) that,

$$[2\phi(|\nabla u_n|) + \phi'(|\nabla u_n|)|\nabla u_n|]\nabla u_n \cdot \nabla v \rightarrow [2\phi(|\nabla u|) + \phi'(|\nabla u|)|\nabla u|]\nabla u \cdot \nabla v \text{ a.e. in } \Omega$$

Applying the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow +\infty} \langle I'(u_n) - I'(u), v \rangle = 0.$$

Hence we have $I \in C^1(W_0^{1,\Phi}(\Omega); \mathbb{R})$ which completes the proof of Proposition 3.5. \blacksquare

Proposition 3.6. *Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Then \mathcal{N} is a C^1 -submanifold of $W_0^{1,\Phi}(\Omega)$. In addition, any critical point of $J|_{\mathcal{N}}$ is a critical point of J .*

Proof of Proposition 3.6. Note that by the very definition of γ_u ,

$$\gamma'_u(t) = \langle J'(tu), u \rangle, \quad u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}.$$

Consider the functional $J_t : W_0^{1,\Phi}(\Omega)(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_t(u) = I_t(u) - \int_{\Omega} f(x, tu)u dx,$$

where

$$I_t(u) = \int_{\Omega} \phi(|\nabla(tu)|)|\nabla(tu)|\nabla u dx.$$

Using arguments as in the proof of Proposition 3.5 one shows that $I_t \in C^1$ and

$$\langle I'_t(u), v \rangle = \int_{\Omega} [2\phi(|\nabla(tu)|) + \phi'(|\nabla(tu)|)|\nabla(tu)|] \nabla u \cdot \nabla v dx, \quad u, v \in W_0^{1,\Phi}(\Omega)(\Omega), \quad t \in \mathbb{R}.$$

One also shows that

$$\gamma''_u(t) = \int_{\Omega} [\phi(|\nabla(tu)|) + \phi'(|\nabla(tu)|)|\nabla(tu)|] |\nabla u|^2 - \int_{\Omega} f'(x, tu)u^2 dx.$$

Set

$$R(u) = \langle J'(u), u \rangle, \quad u \in W_0^{1,\Phi}(\Omega)(\Omega).$$

It follows that $R \in C^1$, (see Proposition 3.5). Actually since $t = 1$ is the global maximum of γ_u , see Proposition 3.2 (i) and Proposition 3.4, we observe that

$$\langle R'(u), u \rangle = \gamma''_u(1) < 0, \quad u \in \mathcal{N}.$$

Using the fact that $\mathcal{N} = R^{-1}(0)$ and 0 is a regular value for R , the set \mathcal{N} is a C^1 -submanifold of $W_0^{1,\Phi}(\Omega)$.

To finish the proof, we assume that $u \in \mathcal{N}$ is a critical point of $J|_{\mathcal{N}}$. Applying the Lagrange Multiplier Theorem, we have

$$J'(u) = \mu R'(u) \quad \text{for some } \mu \in \mathbb{R}.$$

Taking u as a test function it follows that

$$\mu \langle R'(u), u \rangle = \langle J'(u), u \rangle = 0.$$

Reminding that $\langle R'(u), u \rangle = \gamma_u''(1) < 0$ for $u \in \mathcal{N}$ we infer that $\mu = 0$. Therefore $J'(u) \equiv 0$, so that u is a free critical point of J . This completes the proof. \blacksquare

Proposition 3.7. *Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Then there is a constant $C > 0$ such that $\|u\| \geq C$ for each $u \in \mathcal{N}$.*

Proof of Proposition 3.7. Assuming the contrary, there is $(u_n) \subset \mathcal{N}$ such that $\|u_n\| \leq \frac{1}{n}$ for each integer $n \geq 1$. Let $\epsilon > 0$. Using (3.4) and the Poincaré inequality we find some $C_\epsilon > 0$ such that

$$\begin{aligned} \int_{\Omega} \Phi(|\nabla u_n|) dx &\leq \frac{1}{\ell} \int_{\Omega} \phi(|\nabla u_n|) |\nabla u_n|^2 dx \\ &= \frac{1}{\ell} \int_{\Omega} f(x, u_n) u_n dx \leq \frac{\lambda_1 - \epsilon}{\lambda_1} \int_{\Omega} \Phi(|\nabla u_n|) dx + C_\epsilon \int_{\Omega} \Psi(u) dx \end{aligned}$$

Hence

$$\left(1 - \frac{\lambda_1 - \epsilon}{\lambda_1}\right) \int_{\Omega} \Phi(|\nabla u_n|) dx \leq C_\epsilon \int_{\Omega} \Psi(u) dx$$

so that

$$\int_{\Omega} \Phi(|\nabla u_n|) dx \leq \left(1 - \frac{\lambda_1 - \epsilon}{\lambda_1}\right)^{-1} C_\epsilon \int_{\Omega} \Psi(u) dx.$$

Applying Proposition 2.1 we find

$$\|u_n\|^m \leq \int_{\Omega} \Phi(|\nabla u_n|) dx \leq C_\epsilon \max\left(\|u_n\|^{m_\Psi}, \|u_n\|^{\ell_\Psi}\right) = C_\epsilon \|u_n\|^{\ell_\Psi}.$$

Dividing the last expression by $\|u_n\|^m$ we get to

$$1 \leq C_\epsilon \|u_n\|^{\ell_\Psi - m}.$$

Passing to the limit as $n \rightarrow \infty$ and using the fact that $\ell_\Psi > m$ we get to a contradiction. So the Nehari manifold \mathcal{N} is bounded away from zero by some positive constant C . This ends the proof. \blacksquare

4 Proof of Theorem 3.1

At first we will establish a few Lemmas. Set

$$c_{\mathcal{N}} := \inf_{\mathcal{N}} J.$$

The first lemma establishes that any minimizing sequence is bounded in $W_0^{1,\Phi}(\Omega)$.

Lemma 4.1. Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence of J over the Nehari manifold \mathcal{N} , that is, $(u_n) \subset \mathcal{N}$ satisfies $J(u_n) \rightarrow c_{\mathcal{N}}$. Then (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$.

Proof of Lemma 4.1. Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence, that is, $(u_n) \subset \mathcal{N}$ and $J(u_n) \rightarrow c_{\mathcal{N}}$. Assume on the contrary that $\|u_n\| \rightarrow \infty$.

Set $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and there is $v \in W_0^{1,\Phi}(\Omega)$ such that $v_n \rightharpoonup v$ in $W_0^{1,\Phi}(\Omega)$.

We claim that $v \neq 0$. Assume, by the way of contradiction, that $v \equiv 0$. Since $u_n \in \mathcal{N}$ it follows that

$$J(u_n) = \max_{t>0} J(tu_n) \quad \text{for each } n.$$

Let $M > 0$ be a constant. Now we observe that

$$c_{\mathcal{N}} + o_n(1) = J(u_n) \geq J(Mv_n) = \int_{\Omega} \Phi(|\nabla(Mv_n)|)dx - \int_{\Omega} F(x, Mv_n)dx.$$

Since $v_n \rightharpoonup 0$ in $W_0^{1,\Phi}(\Omega)$ it follows by Proposition 3.1 (i) that

$$\int_{\Omega} F(x, Mv_n)dx \rightarrow 0.$$

Employing Proposition 2.1 we have

$$c_{\mathcal{N}} + o_n(1) \geq J(u_n) = \int_{\Omega} \Phi(|\nabla(Mv_n)|)dx + o_n(1) \geq \min(M^{\ell}, M^m) + o_n(1), \quad M > 0.$$

Passing to the limit in the inequalities just above we get

$$c_{\mathcal{N}} \geq \min(M^{\ell}, M^m), \quad M > 0,$$

which is impossible. Therefore $v \neq 0$.

Remember we are assuming that $\|u_n\| \rightarrow \infty$ and $J(u_n) \rightarrow c_{\mathcal{N}}$. Hence

$$\frac{J(u_n)}{\|u_n\|^m} = o_n(1).$$

Applying Proposition 2.1 we have

$$\begin{aligned} \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^m} dx &= \frac{1}{\|u_n\|^m} \int_{\Omega} \Phi(|\nabla u_n|)dx + o_n(1) \\ &\leq \int_{\Omega} \Phi(|\nabla v_n|)dx + o_n(1) = 1 + o_n(1) \end{aligned}$$

Passing to the limit above we have

$$\limsup_{n \rightarrow \infty} \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^m} dx \leq 1$$

On the other hand, it follows by (f_3) and L'Hospital rule that

$$\lim_{t \rightarrow \infty} \frac{F(x, t)}{t^m} = +\infty.$$

Applying Fatou's Lemma and using the fact that $v \neq 0$, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^m} dx &\geq \int_{\Omega} \liminf_{n \rightarrow \infty} \left\{ \frac{F(x, u_n)}{\|u_n\|^m} \right\} dx \\ &= \int_{\Omega} \liminf_{n \rightarrow \infty} \left\{ \frac{F(x, u_n)}{|u_n|^m} |v_n|^m \right\} dx = +\infty, \end{aligned}$$

which is impossible. Thus (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. The proof is complete. ■

Lemma 4.2. Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Then there exists $u \in \mathcal{N}$ such that

$$c_{\mathcal{N}} = J(u) > 0.$$

Proof of Lemma 4.2. Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence for J over \mathcal{N} . By Lemma 4.1, there is $u \in W_0^{1,\Phi}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,\Phi}(\Omega).$$

Claim. $u \not\equiv 0$.

Indeed, assume on the contrary that $u \equiv 0$. Using condition (ϕ_3) we obtain

$$(\ell - 2)(s\phi(s))' \leq (s\phi(s))''s \leq (m - 2)(s\phi(s))'.$$

Integrating from 0 to t in the inequalities above, term by term, we get to

$$(\ell - 2)t\phi(t) \leq (t\phi(t))'t - t\phi(t) \leq (m - 2)t\phi(t).$$

Now we get

$$(\ell - 1)t\phi(t) \leq (t\phi(t))'t \leq (m - 1)t\phi(t),$$

which gives

$$(\ell - 1)t\phi(t) \leq (t\phi(t))'t \leq (m - 1)t\phi(t).$$

Applying arguments like in [11] we get to

$$\ell \leq \frac{t^2\phi(t)}{\Phi(t)} \leq m,$$

which gives

$$\Phi(t) \leq \frac{1}{\ell}t^2\phi(t).$$

Using the inequality just above, the fact that $u_n \in \mathcal{N}$ we have

$$0 \leq \int_{\Omega} \Phi(|\nabla u_n|)dx \leq \frac{1}{\ell} \int_{\Omega} \phi(|\nabla u_n|)|\nabla u_n|^2dx = \frac{1}{\ell} \int_{\Omega} f(x, u_n)u_n. \quad (4.1)$$

Applying Proposition 3.1 (ii), we get

$$\int_{\Omega} f(x, u_n)u_n dx = o_n(1).$$

As a consequence of (4.1), $\|u_n\| \rightarrow 0$, contradicting Proposition 3.7. Therefore $u \not\equiv 0$, proving the **Claim**.

As a consequence of Propositions 3.3, 3.1 (ii) and (3.3) (i), we have

$$u \in W_0^{1,\Phi}(\Omega) \mapsto \langle J'(u), u \rangle \text{ is w.s.l.s.c..}$$

Hence

$$\langle J'(u), u \rangle \leq \liminf_{n \rightarrow \infty} \langle J'(u_n), u_n \rangle = 0.$$

Recall that $\gamma'_u(1) = \langle J'(u), u \rangle \leq 0$. By Proposition 3.4 and its proof there is $t \in (0, 1]$ such that $\gamma'_u(tu) = 0$. Hence $tu \in \mathcal{N}$.

We claim that $t = 1$ so that u is in \mathcal{N} .

Indeed, assume on the contrary, that $t \in (0, 1)$. In this case we get

$$\begin{aligned} c_{\mathcal{N}} &\leq J(tu) = J(tu) - \frac{1}{m} \langle J'(tu), tu \rangle \\ &= \int_{\Omega} \Phi(|\nabla(tu)|) - \frac{1}{m} \phi(|\nabla(tu)|) |\nabla(tu)|^2 dx + \int_{\Omega} \left\{ \frac{1}{m} f(x, tu) tu - F(x, tu) \right\}. \end{aligned} \quad (4.2)$$

Using (f_1) , we get

$$f'(x, t)t - \frac{1}{m} f(x, t) > 0, \quad t > 0.$$

But the inequality above implies that

$$t \mapsto \frac{1}{m} f(x, t)t - F(x, t) \text{ is increasing in } (0, \infty) \text{ for each } x \in \Omega. \quad (4.3)$$

Indeed, using (f_1) we have

$$\frac{d}{dt} \left\{ \frac{1}{m} f(x, t)t - F(x, t) \right\} = t^m \frac{d}{dt} \left\{ \frac{f(x, t)}{t^{m-1}} \right\}, \quad t > 0, x \in \Omega.$$

We also have that

$$t \mapsto \Phi(|\nabla(tu)|) - \frac{1}{m} \phi(|\nabla(tu)|) |\nabla(tu)|^2 \text{ is increasing on } (0, \infty).$$

Indeed, setting

$$L_1(t) = m\Phi(t) - t^2\phi(t), \quad t > 0$$

we find that

$$L_1'(t) = (m-1)t\phi(t) - t(t\phi(t))'.$$

Now we observe that by (ϕ_3) ,

$$(\ell-1)\phi(t) \leq (t\phi(t))' \leq (m-1)\phi(t), \quad t > 0.$$

This shows that L_1 is increasing.

At this point, using (4.2) and (4.3) we conclude that

$$c_{\mathcal{N}} < \int_{\Omega} \left\{ \Phi(|\nabla u|) - \frac{1}{m} \phi(|\nabla u|) |\nabla u|^2 \right\} dx + \int_{\Omega} \left\{ \frac{1}{m} f(x, u)u - F(x, u) \right\} dx.$$

Furthermore, by Proposition 3.3,

$$u \in W_0^{1, \Phi}(\Omega) \mapsto \int_{\Omega} (\Phi(|\nabla u|) - \frac{1}{m} \phi(|\nabla u|) |\nabla u|^2) dx \text{ is w.l.s.c.}$$

Now using once more that

$$u \mapsto \int_{\Omega} F(x, u) dx, \quad u \mapsto \int_{\Omega} f(x, u)u dx \text{ are weakly continuous}$$

we conclude that

$$\begin{aligned} c_{\mathcal{N}} &< \lim_{n \rightarrow \infty} \int_{\Omega} \Phi(|\nabla u_n|) - \frac{1}{m} \phi(|\nabla u_n|) |\nabla u_n|^2 dx + \int_{\Omega} \left\{ \frac{1}{m} f(x, u_n)u_n - F(x, u_n) \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ J(u_n) - \frac{1}{m} J'(u_n)u_n \right\} = c_{\mathcal{N}}, \end{aligned}$$

impossible. Thus $t = 1$ and $u \in \mathcal{N}$. This finishes the proof. \blacksquare

Proof of Theorem 3.1 (conclusion). Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence for J over \mathcal{N} . By the proof of Lemma 4.2 there is $u \in \mathcal{N} \subset W_0^{1,\Phi}(\Omega)$ such that $u_n \rightharpoonup u$.

Claim. $u_n \rightarrow u$ in $W_0^{1,\Phi}(\Omega)$.

Assume the **Claim** has been proved. Since $J \in C^1$ it follows that $J'(u_n) \rightarrow J'(u)$.

By Lemma 4.2, $u \in \mathcal{N}$ and

$$c_{\mathcal{N}} = J(u) = \min_{\mathcal{N}} J > 0.$$

By Proposition 3.6 the set \mathcal{N} is a C^1 -submanifold of $W_0^{1,\Phi}(\Omega)$ so that u is a critical point of $J|_{\mathcal{N}}$ and yet by Proposition 3.6 u is a critical point of J .

Proof of the Claim Let (u_n) be the minimizing sequence for $c_{\mathcal{N}}$. Applying the compactness of the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi}(\Omega)$, it follows that $u_n \rightarrow u$ in $L_{\Phi}(\Omega)$.

Arguing by contradiction, there is $\delta > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \Phi(|\nabla u_n - \nabla u|) dx \geq \delta > 0. \quad (4.4)$$

By the Brézis-Lieb Lemma for convex functions, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \Phi(|\nabla u_n|) - \Phi(|\nabla u_n - \nabla u|) dx = \int_{\Omega} \Phi(|\nabla u|) dx. \quad (4.5)$$

Using (4.4) and (4.5) we infer that

$$\int_{\Omega} \Phi(|\nabla u|) \leq \lim_{n \rightarrow \infty} \int_{\Omega} \Phi(|\nabla u_n|) dx - \delta < \lim_{n \rightarrow \infty} \int_{\Omega} \Phi(|\nabla u_n|) dx.$$

As a consequence, we get by using the Lebesgue Theorem that

$$c_{\mathcal{N}} = \lim_{n \rightarrow \infty} J(u_n) = \lim_{n \rightarrow \infty} \left\{ \int_{\Omega} \Phi(|\nabla u_n|) dx - \int_{\Omega} F(x, u_n) dx \right\} > J(u)$$

which is impossible because of $c_{\mathcal{N}} = \lim_{n \rightarrow \infty} J(u_n) = J(u)$. Therefore $u_n \rightarrow u$ in $W_0^{1,\Phi}(\Omega)$. This finishes the proof. \blacksquare

5 Proof of Theorem 1.1

Consider the two auxiliary functions

$$f^+(x, t) = \begin{cases} f(x, t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0 \end{cases}$$

and

$$f^-(x, t) = \begin{cases} f(x, t) & \text{if } t \leq 0, \\ 0 & \text{if } t > 0. \end{cases}$$

The associated functionals are $J_{\pm} : W_0^{1,\Phi}(\Omega) \rightarrow \mathbb{R}$ given by

$$J_{\pm}(u) = \int_{\Omega} \Phi(|\nabla u|) dx - \int_{\Omega} F^{\pm}(x, u) dx, u \in W_0^{1,\Phi}(\Omega)$$

where $F^\pm(x, t) = \int_0^t f^\pm(x, s)dx, x \in \Omega, t \in \mathbb{R}$. Recalling definitions in Section 3 we see that the Nehari manifolds for the functions f^+ and f^- are respectively,

$$\mathcal{N}^\pm = \{u \in W_0^{1,\Phi}(\Omega) \setminus \{0\} \mid \langle J'_\pm(u), u \rangle = 0\}$$

By Proposition 3.7, \mathcal{N}^\pm are C^1 -manifolds which are away from zero. In addition,

$$c^\pm = \inf_{v \in \mathcal{N}^\pm} J_\pm(v)$$

are critical values of J_\pm . So we obtain two critical points say $u_1, u_2 \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$ such that

$$J_+(u_1) = c^+ > 0 \text{ and } J_-(u_2) = c^- > 0.$$

Given $u \in W_0^{1,\Phi}(\Omega)$ set $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$ so that $u = u^+ + u^-$.

Using u_1^- as a test function we have

$$0 \leq \int_\Omega \Phi(|\nabla u_1^-|) \leq \frac{1}{\ell} \int_\Omega \phi(|\nabla u_1^-|) |\nabla u_1^-|^2 dx = \frac{1}{\ell} \int_\Omega f^+(x, u_1) u_1^- dx = 0. \quad (5.1)$$

As a consequence of (5.1) one has $u_1^- \equiv 0$ and so $u_1 = u_1^+ \geq 0$ in Ω . Similarly, we also obtain $u_2 = u_2^- \leq 0$. Now, since \mathcal{N}^\pm are away from zero, it follows that $u_1, u_2 \neq 0$. Hence by the Maximum Principle, (cf. Pucci & Serrin [25]), $u_1 > 0$ and $u_2 < 0$ in Ω . For further comments in this regard we refer the reader to Carvalho et al [7].

We add that the functions u_1, u_2 are critical points of J . So u_1, u_2 are also weak solutions of problem (1.1).

In what follows we shall prove that problem (1.1) admits at least one sign changing solution. At first, we define the nodal Nehari manifold by

$$\mathcal{N}_{nod} = \{u \in W_0^{1,\Phi}(\Omega) \setminus \{0\} \mid u^\pm \neq 0, \langle J'(u), u^\pm \rangle = 0\}.$$

Consider the nodal level given by

$$c_{nod} = \inf_{v \in \mathcal{N}_{nod}} J(v).$$

It is easy to verify that any sign changing solution for the problem (1.1) should belong to \mathcal{N}_{nod} . Hence it is natural to consider the nodal Nehari manifold in order to ensure the existence of sign changing solutions.

So, our aim is to prove that the Nehari manifold \mathcal{N}_{nod} is not empty.

In this regard, consider the function $\theta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by

$$\theta(t, s) := J(tu^+ + su^-)$$

where $u \in W_0^{1,\Phi}(\Omega)$ and $u^\pm \neq 0$. Given $t, s > 0$ we emphasize that $\nabla \theta(t, s) = 0$ if only if $tu^+ + su^- \in \mathcal{N}_{nod}$. In other words, the critical points of the function θ provide us with elements on the Nehari nodal set \mathcal{N}_{nod} .

Given $u \in W_0^{1,\Phi}(\Omega)$ with $u^\pm \neq 0$, using once more the fibering maps, we will prove that $tu^+ + su^- \in \mathcal{N}_{nod}$ for some suitable numbers $t, s \in (0, \infty)$. We need a technical result which we state and prove below.

Proposition 5.1. Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Let $u \in W_0^{1,\Phi}(\Omega)$ with $u^\pm \neq 0$. Then

$$\lim_{t \rightarrow \infty, s \rightarrow \infty} \frac{\theta(t, s)}{t^m + s^m} = -\infty.$$

and

$$\lim_{t \rightarrow 0, s \rightarrow 0} \frac{\theta(t, s)}{t^m + s^m} > 0.$$

Proof of Proposition 5.1. Let $t, s > 0$. We have

$$\begin{aligned} \theta(t, s) &= \int_{\Omega} \Phi(|\nabla tu^+|) dx - \int_{\Omega} F(x, tu^+) dx \\ &+ \int_{\Omega} \Phi(|\nabla su^-|) dx - \int_{\Omega} F(x, su^-) dx \\ &\geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \int_{\Omega} \Phi(|\nabla tu^+|) dx + C_\varepsilon \int_{\Omega} \Psi(tu^+) dx \\ &+ \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \int_{\Omega} \Phi(|\nabla su^-|) dx + C_\varepsilon \int_{\Omega} \Psi(su^-) dx. \end{aligned}$$

Applying Proposition 2.1 twice we get to

$$\begin{aligned} \theta(t, s) &\geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) t^m \int_{\Omega} \Phi(|\nabla u^+|) dx - C_\varepsilon t^{\ell_\Psi} \int_{\Omega} \Psi(u^+) dx + \\ &\quad \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) s^m \int_{\Omega} \Phi(|\nabla u^-|) dx - C_\varepsilon s^{\ell_\Psi} \int_{\Omega} \Psi(u^-) dx \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\theta(t, s)}{t^m + s^m} &\geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \min \left[\int_{\Omega} \Phi(|\nabla u^+|) dx, \int_{\Omega} \Phi(|\nabla u^-|) dx \right] \\ &- C_\varepsilon \frac{\max(t^{\ell_\Psi}, s^{\ell_\Psi})}{t^m + s^m} \int_{\Omega} \Psi(u) dx. \end{aligned}$$

As a consequence,

$$\lim_{t \rightarrow 0, s \rightarrow 0} \frac{\theta(t, s)}{t^m + s^m} \geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \min \left[\int_{\Omega} \Phi(|\nabla u^+|) dx, \int_{\Omega} \Phi(|\nabla u^-|) dx \right] > 0$$

for each $\varepsilon > 0$ small enough.

Next we shall compute the limit at infinity of $\theta(t, s)/(t^m + s^m)$. By Proposition 2.1,

$$\theta(t, s) \leq t^m \int_{\Omega} \Phi(|\nabla u^+|) dx + s^m \int_{\Omega} \Phi(|\nabla u^-|) dx - \int_{\Omega} F(x, tu^+ + su^-) dx$$

which yields

$$\frac{\theta(t, s)}{t^m + s^m} \leq \max \left(\int_{\Omega} \Phi(|\nabla u^+|) dx, \int_{\Omega} \Phi(|\nabla u^-|) dx \right) - \frac{1}{t^m + s^m} \int_{\Omega} F(x, tu^+ + su^-) dx$$

for $t, s \geq 1$. As a consequence, we have

$$\begin{aligned} \lim_{t \rightarrow \infty, s \rightarrow \infty} \frac{\theta(t, s)}{t^m + s^m} &\leq \max \left(\int_{\Omega} \Phi(|\nabla u^+|) dx, \int_{\Omega} \Phi(|\nabla u^-|) dx \right) \\ &\quad - \liminf_{t \rightarrow \infty, s \rightarrow \infty} \int_{\Omega} \frac{F(x, tu^+ + su^-)}{t^m + s^m} dx. \end{aligned} \quad (5.2)$$

We have,

$$\liminf_{t \rightarrow \infty, s \rightarrow \infty} \int_{\Omega} \frac{F(x, tu^+ + su^-)}{t^m + s^m} dx \geq \liminf_{t \rightarrow \infty, s \rightarrow \infty} \int_{\Omega} \frac{F(x, tu^+ + su^-)}{|tu^+ + su^-|^m} \frac{|tu^+ + su^-|^m}{t^m + s^m} dx$$

By Fatou's Lemma and (f_3) it follows that

$$\begin{aligned} \infty &= \liminf_{t \rightarrow \infty, s \rightarrow \infty} \int_{\Omega} \frac{F(x, tu^+ + su^-)}{|tu^+ + su^-|^m} \min(|u^+|^m, |u^-|^m) dx \\ &\leq \liminf_{t \rightarrow \infty, s \rightarrow \infty} \int_{\Omega} \frac{F(x, tu^+ + su^-)}{t^m + s^m} dx. \end{aligned} \quad (5.3)$$

It follows from (5.2) and (5.3) that

$$\lim_{t \rightarrow \infty, s \rightarrow \infty} \frac{\theta(t, s)}{t^m + s^m} = -\infty.$$

This ends the proof. ■

Proposition 5.2. Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Let $u \in W_0^{1,\Phi}(\Omega)$ such that $u^{\pm} \neq 0$. Then there exist uniquely determined $t, s \in (0, \infty)$ such that $tu^+ + su^- \in \mathcal{N}_{nod}$.

Proof of Proposition 5.2. By Proposition 5.1 we infer that

$$\lim_{t \rightarrow \infty, s \rightarrow \infty} \frac{\theta(t, s)}{t^m + s^m} = -\infty.$$

and

$$\lim_{t \rightarrow 0, s \rightarrow 0} \frac{\theta(t, s)}{t^m + s^m} > 0.$$

Since $\theta(t, s)$ is continuous, there exist $t_0, s_0 \in (0, \infty)$ such that

$$\max_{t \geq 0, s \geq 0} \theta(t, s) = \theta(t_0, s_0).$$

By Proposition 3.4 and

$$\frac{\partial \theta(t_0, s_0)}{\partial t} = \gamma'_{u^+}(t_0) = 0, \quad \frac{\partial \theta(t_0, s_0)}{\partial s} = \gamma'_{u^+}(s_0) = 0$$

it follows that t_0, s_0 are uniquely determined. By an easy argument one shows that $t_0 u^+ + s_0 u^- \in \mathcal{N}_{nod}$. This ends the proof. ■

Proposition 5.3. Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Let $(u_n) \subseteq \mathcal{N}_{nod}$ be a minimizing sequence for J . Then (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$.

Proof of Proposition 5.3. The proof follows the same lines as in the proof of Lemma 4.1. Since $u_n = u_n^+ + u_n^-$ it suffices to show that (u_n^+) and (u_n^-) are bounded in $W_0^{1,\Phi}(\Omega)$. We skip the details. ■

Proposition 5.4. Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Then there is $u \in \mathcal{N}_{nod}$ such that $J(u) = c_{nod}$.

Proof of Proposition 5.4. Let $(u_n) \subseteq \mathcal{N}_{nod}$ be a minimizing sequence for J over \mathcal{N}_{nod} , that is

$$J(u_n) \rightarrow c_{nod}.$$

We split u_n as $u_n = u_n^+ + u_n^-$. By Proposition 5.3, (u_n) is bounded in the reflexive space $W_0^{1,\Phi}(\Omega)(\Omega)$ and as a consequence, both (u_n^+) and (u_n^-) are also bounded in $W_0^{1,\Phi}(\Omega)(\Omega)$ and by the very definition of \mathcal{N}_{nod} , $u_n^\pm \neq 0$. Thus,

$$\begin{aligned} u_n &\rightharpoonup u, \quad u_n^+ \rightharpoonup u^+, \quad u_n^- \rightharpoonup u^- \text{ in } W_0^{1,\Phi}(\Omega)(\Omega), \\ u^+ \cdot u^- &= 0, \quad \text{supp}(u_n^+) \cap \text{supp}(u_n^-) = \emptyset, \\ \langle J'(u_n), u_n \rangle &= \langle J'(u_n^+), u_n^+ \rangle = \langle J'(u_n^-), u_n^- \rangle = 0 \end{aligned}$$

so that, in particular, $(u_n^\pm) \subseteq \mathcal{N}$.

Claim. $u^+, u^- \neq 0$.

Indeed, assume on the contrary that $u^+ = 0$. Using the fact that $u_n^+ \in \mathcal{N}$ we have

$$\begin{aligned} 0 \leq \int_{\Omega} \Phi(|\nabla u_n^+|) dx &\leq \frac{1}{\ell} \int_{\Omega} \phi(|\nabla u_n^+|) |\nabla u_n^+|^2 dx \\ &= \frac{1}{\ell} \int_{\Omega} f(x, u_n^+) u_n^+ dx \\ &= \frac{1}{\ell} o_n(1). \end{aligned}$$

It follows that

$$\min\{\|u_n^+\|^\ell, \|u_n^+\|^m\} \leq \int_{\Omega} \Phi(|\nabla u_n^+|) dx = o_n(1)$$

and so $\|u_n^+\| \rightarrow 0$. Assuming that $u^- = 0$ we find in a similar way that $\|u_n^-\| \rightarrow 0$. By Proposition 3.7 and the fact that $u_n^\pm \rightharpoonup u^\pm$ in $W_0^{1,\Phi}(\Omega)(\Omega)$ we have

$$0 < C \leq \|u^\pm\| \leq \liminf \|u_n^\pm\|$$

which is impossible. This proves the **Claim**.

By Proposition 5.2 there exist $t, s \in (0, \infty)$ such that $tu^+ + su^- \in \mathcal{N}_{nod}$. Set

$$\tilde{u} = tu^+ + su^-.$$

At this point we have

$$tu^+, su^- \neq 0, \quad tu^+ \cdot su^- = 0, \quad \text{supp}(tu^+) \cap \text{supp}(su^-) = \emptyset. \quad (5.4)$$

Using the fact that $\tilde{u} = tu^+ + su^- \in \mathcal{N}_{nod}$ and (5.4) we have

$$0 = \langle J'(\tilde{u}), \tilde{u}^\pm \rangle = \langle J'(tu^+), tu^+ \rangle = \langle J'(su^-), su^- \rangle.$$

On the other hand, using Proposition 3.3 (i),

$$\langle J'(u^+), u^+ \rangle \leq \liminf \langle J'(u_n^+), u_n^+ \rangle = 0$$

and

$$\langle J'(u^-), u^- \rangle \leq \liminf \langle J'(u_n^-), u_n^- \rangle = 0.$$

As a consequence of the inequalities just above, $0 < t, s \leq 1$, because if otherwise, one of t, s , say $t > 1$ we would have $\langle J'(\tau u^+), \tau u^+ \rangle > 0$ for each $\tau \in (0, t)$. But since $\gamma'_{u^+}(1) \leq 0$ we have a contradiction. Now,

$$\begin{aligned} c_{nod} \leq J(\tilde{u}) &= J(tu^+ + su^-) = J(tu^+) + J(su^-) \\ &\leq \liminf J(tu_n^+) + \liminf J(su_n^-) \\ &\leq \liminf (J(tu_n^+ + su_n^-)) \\ &\leq \liminf \max_{t,s>0} \theta(t, s) \end{aligned}$$

But, by the proof of Proposition 5.2, we have $\max_{t>0, s>0} \theta(t, s) = \theta(1, 1)$. Hence,

$$\begin{aligned} c_{nod} &\leq \liminf \theta(1, 1) \\ &= \liminf J(u_n^+ + u_n^-) = c_{nod}. \end{aligned}$$

This ends the proof of Proposition 5.4. ■

Next we shall prove that any minimizer for J on \mathcal{N}_{nod} is a free critical point.

Proposition 5.5. *Assume $(\phi_1) - (\phi_3)$, $(f_0) - (f_3)$. Let $u \in \mathcal{N}_{nod}$ be a minimizer for J over \mathcal{N}_{nod} , that is $J(u) = c_{nod}$. Then u is a free critical point for J .*

Proof of Proposition 5.5. We shall adapt arguments by Szulkin & Weth in [29] to the case of the Φ -Laplace operator. For the reader's convenience, we shall provide a few details.

Remind that $\gamma_{u^+}(1)$ and $\gamma_{u^-}(1)$ are the unique maximum values for the functions $\gamma_{u^+}, \gamma_{u^-}$, respectively. We have

$$\begin{aligned} J(su^+ + tu^-) &= J(su^+) + J(tu^-) \\ &< J(u^+) + J(u^-) \\ &= J(u) \\ &= c_{nod}, \quad s, t \geq 0, \quad s, t \neq 1. \end{aligned} \tag{5.5}$$

In order to show that $u \in \mathcal{N}_{nod}$ is a critical point for J , assume by the way of contradiction that $J'(u) \neq 0$.

Then there exist $\delta, \mu > 0$ such that

$$\|J'(v)\| \geq \mu \quad \text{whenever} \quad \|v - u\| \leq 3\delta, \quad v \in W_0^{1,\Phi}(\Omega).$$

Let $D = D_0 \times D_0$ where $D_0 = (\frac{1}{2}, \frac{2}{3})$. Consider the continuous function $g : \overline{D} \rightarrow W_0^{1,\Phi}(\Omega)$ defined by $g(s, t) = tu^+ + su^-$. Note that $t \rightarrow \gamma_{u^+}(t)$ and $t \rightarrow \gamma_{u^-}(t)$ are strictly increasing functions on $[0, 1]$. It follows that

$$J(g(s, t)) = J(su^+ + tu^-) = c_{nod} \text{ if and only if } t = s = 1.$$

Moreover, due the estimates in (5.5), we have $J(g(s, t)) < c_{nod}$ for $t, s > 1$. Hence

$$\beta := \max_{\partial D} J(g(s, t)) < c_{nod}.$$

By the Deformation Lemma (cf. [33, Lemma 2.3]) with $\epsilon := \min \left\{ \frac{c_{nod} - \beta}{4}, \frac{\mu\delta}{8} \right\}$ there is $\eta \in C \left([0, 1] \times W_0^{1,\Phi}(\Omega), W_0^{1,\Phi}(\Omega) \right)$ such that

1. $\eta(1, v) = v$ if $v \notin J^{-1}([c_{nod} - 2\epsilon, c_{nod} + 2\epsilon])$,
2. $J(\eta(1, v)) \leq c_{nod} - \epsilon$, for each $v \in W_0^{1,\Phi}(\Omega)$ such that $\|v - u\| \leq \delta$ and $J(v) \leq c_{nod} + \epsilon$,
3. $J(\eta(1, v)) \leq J(v)$ for each $v \in W_0^{1,\Phi}(\Omega)(\Omega)$.

Since the maximum value of $J \circ g$ is achieved at $(s, t) = (1, 1)$ it follows by item (3) just above that

$$\begin{aligned} \max_{(s,t) \in D} J(\eta(1, g(s, t))) &\leq \max_{(s,t) \in D} J(g(s, t)) \\ &< c_{nod} \\ &= \max_{\{(s,t) \in [0,1] \times [0,1]\}} J(g(s, t)), \end{aligned}$$

so that $\eta(1, g(s, t)) \notin \mathcal{N}_{nod}$ for each $(s, t) \in D$.

Now we claim that there is $(s_0, t_0) \in D$ such that $\eta(1, g(s_0, t_0)) \in \mathcal{N}_{nod}$ which will lead to a contradiction, showing that u is a critical point of J .

The proof of the claim will be achieved using degree theory. Consider the maps

$$h(s, t) := \eta(1, g(s, t)) = \eta(1, su^+ + tu^-),$$

$$\Psi_0(s, t) = (\langle J'(su^+), u^+ \rangle, \langle J'(su^-), u^- \rangle)$$

and

$$\Psi_1(s, t) := \left(\frac{1}{s} \langle J'(h^+(s, t)), h^+(s, t) \rangle, \frac{1}{t} \langle J'(h^-(s, t)), h^-(s, t) \rangle \right).$$

We claim that $\deg(\Psi_0, D, (0, 0)) = 1$. Indeed,

$$\langle J'(su^\pm), u^\pm \rangle > 0, \quad s \in (0, 1), \quad \text{due the fact that } \gamma_{u^\pm}(s) > 0 \text{ for } s \in (0, 1)$$

and

$$\langle J'(su^\pm), u^\pm \rangle < 0, \quad s \in (1, +\infty) \quad \text{because } \gamma_{u^\pm}(s) < 0 \text{ for } s \in (1, +\infty).$$

Now, consider the homotopy

$$H(\tau, s) = (1 - \tau)L(s) + \tau s,$$

where $L(s) := \langle J'(su^+), u^+ \rangle$.

Since $H(\tau, s) \neq 0$ for each $\tau \in [0, 1]$ and $s \in \{\frac{1}{2}, \frac{2}{3}\}$, we have by the homotopy invariance property that

$$\deg(L, D_0, 0) = \deg(\text{Id}_{\mathbb{R}}, D_0, 0) = 1.$$

By product formula for the degree the claim holds.

Note that

$$J(g(s, t)) \leq \beta = \max_{\partial D} J \circ g \leq c_{nod} - 4\epsilon < c_{nod} - 2\epsilon.$$

Hence $g(s, t) \notin J^{-1}([c_{nod} - 2\epsilon, c_{nod} + 2\epsilon])$. So, by item (1) above we see that

$$h(s, t) := \eta(1, g(s, t)) = g(s, t), \quad (s, t) \in \partial D.$$

Now, note that

$$\begin{aligned} \Psi_1(s, t) &= \left(\frac{1}{s} \langle J'(g^+(s, t)), g^+(s, t) \rangle, \langle \frac{1}{t} J'(g^-(s, t)), g^-(s, t) \rangle \right) \\ &= (\langle J'(su^+), u^+ \rangle, \langle J'(tu^-), u^- \rangle) \\ &= \Psi_0(s, t), \quad (s, t) \in \partial D, \end{aligned}$$

so that $\Psi_1 = \Psi_0$ on ∂D . This implies that

$$\deg(\Psi_1, D, (0, 0)) = \deg(\Psi_0, D, (0, 0)) = 1.$$

As a consequence there is $(s_0, t_0) \in D$ such that $\Psi_1(s_0, t_0) = (0, 0)$. This is equivalent to

$$\langle J'(h^\pm(s_0, t_0)), h^\pm(s_0, t_0) \rangle = 0,$$

showing that $h^\pm(s_0, t_0) \in \mathcal{N}$. Therefore $\eta(1, g(s_0, t_0)) = h(s_0, t_0) \in \mathcal{N}_{nod}$ which is a contradiction.

Applying Propositions 5.4 and 5.5 ends the proof of Theorem 1.1. ■

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